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A quantum Cerenkov effect

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Abstract. We show that when a point-like, neutral, but polarisable body moves with Cerenkov velocity in a medium whose collective excitations are in the ground state, there is a radiation process.

It is well known that when a quantised field is subject to an external variable potential, there is particle creation (see for instance Reed and Simon (1979)). However, there exist only a few examples amenable to exact calculation (as opposed to perturbation theory). One such example is the accelerated mirror in 2D spacetime (Fulling and Davies 1976) which shows strong similarities with radiation by an accelerated charge and suggests that something akin to Cerenkov radiation could also occur. In this paper we show that this is indeed true, the calculation being also an exact one.

To formulate the problem we shall take a massless scalar field Ψ, we shall work in 4D spacetime (in two dimensions there is no Cerenkov cone) and take a point-like polarisable body, moving with velocity v along the x axis, instead of a mirror; the quantised field Ψ represents photons in a medium with index $n > 1$. The equations describing the system are

$$n^2\partial_t^2\Psi - \nabla^2\Psi = \delta(x - vt)\delta(y)\delta(z)Q(t) \tag{1a}$$

$$Q(t) = \lambda\Psi(vt, 0, 0, t) \tag{1b}$$

where λ is a polarisation coefficient, and $h = c = 1$ units are used throughout.

These equations are a very schematic prototype of more realistic ones with the Maxwell field instead of Ψ and with an electric (or magnetic) frequency dependent susceptibility instead of λ. They are similar (with $n = 1$) to the so-called Wentzel model (Schweber 1962). A possible application (of astrophysical interest) is a grain in a plasma.

We shall proceed as usual in quantum optics: a spatial Fourier expansion in (1a) gives the Heisenberg picture equations for the creation–annihilation operators, the particle interpretation being given in the Fock space of the free field:

$$\partial_t a_K(t) = -\mathrm{i}\omega_K a_K + \frac{\mathrm{i}}{n^2}(2\omega_K)^{-1/2}\exp(\mathrm{i}k_x vt)Q(t) \tag{2}$$

(and its conjugate for $a_K^*(t)$ where $\omega_K = (k_x^2 + k_y^2 + k_z^2)^{1/2}/n$ is the frequency and $a_K(t)$ is the annihilation operator of the mode $\boldsymbol{K}$).

A Laplace transform $\int_0^t \mathrm{e}^{-st}(\cdot)$ then gives:

$$(s+\mathrm{i}\omega_K)a_K(s)=a_K(0)+\frac{\mathrm{i}}{n^2}(2\omega_K)^{-1/2}Q(s-\mathrm{i}k_xv) \tag{3a}$$

where $a_K(0)=a_K(t=0)$ (and the conjugate for $a_K^*(s)$) and $(1b)$ is equivalent to

$$Q(s)=\frac{\lambda}{(2\pi)^3}\int\frac{\mathrm{d}^3k}{(2\omega_K)^{1/2}}[a_K(s+\mathrm{i}k_xv)+a_K^*(s-\mathrm{i}k_xv)]. \tag{3b}$$

Using $(3a)$ in $(3b)$ one gets a simple equation for $Q(s)$: inserting $Q(s\pm\mathrm{i}k_xv)$ in $(3a)$ gives

$$a_K(s)=\frac{a_K(0)}{s+\mathrm{i}\omega_K}+\frac{\mathrm{i}\lambda^2}{(2\pi)^3n^2}\frac{F(s-\mathrm{i}k_xv)}{s+\mathrm{i}\omega_K}$$
$$\times\int\frac{\mathrm{d}^3p}{(2\omega_p)^{1/2}}\left(\frac{a_p(0)}{s+\mathrm{i}(p_x-k_x)v+\mathrm{i}\omega_\mathrm{p}}+\frac{a_p^*(0)}{s-\mathrm{i}(p_x-k_x)v-\mathrm{i}\omega_\mathrm{p}}\right) \tag{4}$$

where:

$$F(s)=\left(1-\frac{\lambda}{(2\pi)^3n^2}\int\frac{\mathrm{d}^3k}{\omega_K}\frac{k_xv+\omega_K}{s^2+(k_xv+\omega_K)^2}\right)^{-1}. \tag{5}$$

In this expression the k integral is supposed to be made convergent by some cut-off taking into account that the polarisable body has a finite structure and that n cannot be constant for any k. From $a_K(s)$ one has the explicit expression of $a_K(t)$ (and $a_K^*(t)$) and various observables can be computed.

Let us start at $t=0$ with the Ψ field ground state: $a_K(0)|0\rangle=0$ (any $\boldsymbol{k}$) and calculate

$$\partial_t\langle 0|N_K(t)|0\rangle=\partial_t\langle 0|a_K^*(t)a_K(t)|0\rangle$$

the rate of increase of the mean number of photons in the mode $\boldsymbol{k}$ of the free field.

Its Laplace transform is

$$s\langle 0|N_K(s)|0\rangle=s\int_\Gamma \mathrm{d}s'\langle 0|a_K^*(s')a_K(s-s')|0\rangle \tag{6}$$

for Re $s\gg 0$ and Γ just at the right of the s' imaginary axis; inserting (4) and using $\langle 0|a_K(0)a_{K'}^*(0)|0\rangle=\delta(K-k')$ gives

$$s\langle 0|N_K(s)|0\rangle=\frac{\lambda^4}{2(2\pi)^6n^4\omega_K}\int_\Gamma \mathrm{d}s'\frac{F(s'+\mathrm{i}k_xv)F(s'+\mathrm{i}k_xv-s)}{(s'-\mathrm{i}\omega_K)(s'-\mathrm{i}\omega_K-s)}$$
$$\times\int\frac{\mathrm{d}^3p}{\omega_\mathrm{p}}\frac{s}{[s'+\mathrm{i}(p_x+k_x)v+\mathrm{i}\omega_p][s'+\mathrm{i}(p_x+k_x)v+\mathrm{i}\omega_p-s]}. \tag{7}$$

In this, the function $F(z)$ is defined for Re $z>0$ but can be continued towards Re $z<0$; depending on the choice of cut-off, some spurious singularities may appear far away; we shall ignore them. It remains the two poles $s'=\mathrm{i}\omega_K$ and $s'=\mathrm{i}\omega_K+s$ and the singularities of the p integral: this integral can be written using spherical variables and $x=|\boldsymbol{P}|/n$ as

$$\pi\int_0^\pi \mathrm{d}\theta\sin\theta\int_0^\infty x\,\mathrm{d}x\left(\frac{1}{s'+\mathrm{i}k_xv+\mathrm{i}x(1+nv\cos\theta)}-\frac{1}{s'+\mathrm{i}k_xv+\mathrm{i}x(1+nv\cos\theta)-s}\right)$$

and one must distinguish two cases.

Case 1. $v<1/n$, taking $z=x(1+nv\cos\theta)$ gives this integral as

$$C\int_0^\infty z\,\mathrm{d}z\left(\frac{1}{s'+\mathrm{i}k_xv+\mathrm{i}z}-\frac{1}{s'+\mathrm{i}k_xv+\mathrm{i}z-s}\right) \tag{8}$$

C being the θ integration result. One sees that this gives logarithmic branch points and it is more convenient to perform the s' integration first; there are only poles and the sum of residues gives

$$\langle 0|sN_K(s)|0\rangle=\frac{C\lambda^4}{2(2\pi)^5n^4\omega_K}F(\mathrm{i}\omega_K+\mathrm{i}k_xv)F(\mathrm{i}\omega_K+\mathrm{i}k_xv-s)$$
$$\times\int_0^\infty \mathrm{d}z\frac{z}{s}\left(\frac{1}{\mathrm{i}(\omega_K+k_xv+z)+s}-\frac{1}{\mathrm{i}(\omega_K+k_xv+z)-s}\right).$$

Again, the inverse Laplace transform can be performed before the z integration to yield:

$$\partial_t\langle 0|N_K(t)|0\rangle=A\int_0^\infty \mathrm{d}z\,z\frac{\sin(\omega_K+k_xv+z)t}{\omega_K+k_xv+z}$$

where all coefficients are packed in A; for $t\to\infty$ this is

$$\partial_t\langle 0|N_K|0\rangle=A\int_0^\infty \mathrm{d}z\,z\delta(\omega_K+k_xv+z)$$

which is null because $\omega_K+k_xv\geqslant 0$.

Case 2. $v>1/n$ (the Cerenkov velocity) in this case $1+nv\cos\theta=0$ for some θ_c on $(0,\pi)$ and instead of (8) one obtains

$$C_1\int_0^\infty \mathrm{d}z\,z\left(\frac{1}{s'+\mathrm{i}k_xv+\mathrm{i}z}-\frac{1}{s'+\mathrm{i}k_xv+\mathrm{i}z-s}\right)$$
$$+C_2\int_0^\infty \mathrm{d}z\,z\left(\frac{1}{s'+\mathrm{i}k_xv-\mathrm{i}z}-\frac{1}{s'+\mathrm{i}k_xv-\mathrm{i}z-s}\right)$$

where C_1, C_2 are the results of the θ integration on $(0,\theta_c-\varepsilon)$, $(\theta_c+\varepsilon,\pi)$ and ε depends on the cut-off chosen. Proceeding as in case 1 one obtains for $t\to\infty$

$$\partial_t\langle 0|N_K(t)|0\rangle=A_1\int_0^\infty \mathrm{d}z\,z\delta(\omega_K+k_xv-z)+A_2\int_0^\infty \mathrm{d}z\,z\delta(\omega_K+k_xv-z).$$

This is non-null and exhibits the Cerenkov radiation. $\partial_t\int \mathrm{d}^3k\langle 0|\omega_KN_K|0\rangle$ could be calculated too, and gives the rate of deceleration due to radiation.

This radiation is very different from the usual one: for a charge e one would have directly $Q(s)=e/s$ instead of (3b) and the same calculation would give $\partial_t\langle 0|N_K(t)|0\rangle=A\delta(\omega_K+k_xv)$ which gives photons only on the Cerenkov cone $\omega_K+k_xv=0$; also the calculation remains the same if Ψ is not quantised. Here on the contrary the radiation is distributed on all modes except those on the cone, and it needs quantisation; it disappears if $h\to 0$. Moreover it is 'chaotic': $\langle 0|\Psi|0\rangle$ remains null, but by analogy with the uniformly accelerated mirror case which gives a thermal field, it is likely that the coherence function (which cannot be calculated simply as N_K) satisfies $g^2>1$ and that it admit a classical representation.

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